

ONE-DIMENSIONAL, NON-LOCAL, FIRST-ORDER, STATIONARY MEAN-FIELD GAMES WITH CONGESTION: A FOURIER APPROACH

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ABSTRACT. Here, we study a one-dimensional, non-local mean-field game model with congestion. When the kernel in the non-local coupling is a trigonometric polynomial we reduce the problem to a finite dimensional system. Furthermore, we treat the general case by approximating the kernel with trigonometric polynomials. Our technique is based on Fourier expansion methods.

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1. INTRODUCTION

In this paper, we consider the following mean-field game (MFG) model

$$\begin{cases} \frac{(u_x+c)^2}{2m^{2-\alpha}} + V(x) = \int_{\mathbb{T}} G(x-y)m(y)dy + \overline{H}, \\ (m^{\alpha-1}(u_x+c))_x = 0, \\ m > 0, \int_{\mathbb{T}} m(x)dx = 1, \end{cases} \quad (1.1)$$

where, $G \in C^2(\mathbb{T})$ is a given kernel, $V \in C^2(\mathbb{T})$ is a given C^2 potential and $0 < \alpha \leq 2$, $c \in \mathbb{R}$ are given parameters. The unknowns are functions $u, m : \mathbb{T} \rightarrow \mathbb{R}$ and the number $\overline{H} \in \mathbb{R}$. We study the existence of smooth solutions for (1.1) and analyze their properties and solution methods.

MFGs theory was introduced by J-M. Lasry and P-L. Lions in [33, 34, 35, 36] and by M. Huang, P. Caines and R. Malhamé in [31, 32] to study large populations of

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agents that play dynamic differential games. Mathematically, MFGs are given by the following system

$$\begin{cases} -u_t(x, t) - \sigma^2 \Delta_x u(x, t) + H(x, D_x u(x, t), m(x, t), t) = 0, \\ m_t(x, t) - \sigma^2 \Delta_x m(x, t) - \operatorname{div}_x (D_p H(x, D_x u(x, t), m(x, t), t)) = 0, \\ m(x, 0) = m^0(x), \quad u(x, T) = u^T(x, m^0(x)). \end{cases} \quad (1.2)$$

where $m(x, t)$ is the distribution of the population at time t , and $u(x, t)$ is the value function of the individual player, and T is the terminal time. Furthermore, $H : \mathbb{T}^d \times \mathbb{R}^d \times X \times \mathbb{R} \rightarrow \mathbb{R}$, $(x, p, m, t) \mapsto H(x, p, m, t)$ is the Hamiltonian of the system, where $X = \mathbb{R}^+$ or $L^1(\mathbb{T}^d; \mathbb{R}^+)$ or $\mathbb{R}^+ \times L^1(\mathbb{T}^d; \mathbb{R}^+)$, and $\sigma \geq 0$ is the diffusion parameter. Finally, m^0, u^T are given initial-terminal conditions.

Suppose $L : \mathbb{T}^d \times \mathbb{R}^d \times X \times \mathbb{R} \rightarrow \mathbb{R}$, $(x, v, m, t) \mapsto L(x, v, m, t)$ is the Legendre transform of H . Then, formally, (1.2) are the optimality conditions for a population of agents where each agent aims to minimize the action

$$u(x, t) = \inf_v \left\{ \mathbb{E} \int_t^T L(x(s), v(s), m(x(s), s), s) ds + u^T(x(T), m(x(T), T)) \right\}, \quad (1.3)$$

where the infimum is taken over all progressively measurable controls $v(s)$, and trajectories $x(s)$ are governed by

$$dx(s) = v(s)ds + \sqrt{2}\sigma dW_s, \quad x(0) = x,$$

for a standard d -dimensional Brownian motion $\{W_s\}$. Assume that are driven by mutually independent Brownian motions.

Indeed, the first equation in (1.2) is the Hamilton-Jacobi equation for the value function u . Furthermore, optimal velocities of agents are given by

$$v(t) = -D_p H(x, D_x u(x(t), t), m(x(t), t)),$$

thus the second equation in (1.2) which is the corresponding Fokker-Planck equation. Rigorous derivations of (1.2) in various contexts can be found in [33, 34, 35, 36, 37, 8, 26] and references therein.

Actions of the total population affect an individual agent through the dependence of H and L on m . The type of the dependence of H and L on m is called the *coupling*, and it can be either local, global or mixed. Spatial preferences of agents are encoded in the x dependence of H and L .

Our problem of interest (1.1) is the 1-dimensional, stationary, first-order version of (1.2) with Hamiltonian

$$H(x, p, m, t) = \frac{(p + c)^2}{2m^{2-\alpha}} + V(x) - \int_{\mathbb{T}} G(x - y)m(y)dy. \quad (1.4)$$

Since seminal papers [33, 34, 35, 36, 31, 32] a substantial amount of research has been done in MFGs. Classical solutions were studied extensively both in stationary and non-stationary settings in [35, 24, 25, 40] and in [20, 19, 22, 21], respectively. Weak solutions were addressed in [41, 42, 9, 10] for time-dependent problems and in [15] for stationary problems. Numerical methods can be found in [5, 1, 4, 2, 3, 12, 7, 29, 6].

Nevertheless, most of the previous work concerns problems where Hamiltonian does not have singularity at $m = 0$. The problems where Hamiltonian has singularity at $m = 0$, such as in (1.4), are called *congestion* problems. The reason is that the

Lagrangian corresponding to H in (1.4) is

$$L(x, v, m, t) = \frac{v^2 m^{2-\alpha}}{2} + cv - V(x) + \int_{\mathbb{T}} G(x - y) m(y) dy,$$

and in the view of (1.3) agents pay high price for moving at high speeds in dense areas.

Congestion problems were previously studied in [38, 16, 27, 28, 17, 30, 39, 11]. Uniqueness of smooth solutions was established in [38]. Existence of smooth solutions for stationary second-order local MFG with quadratic Hamiltonian was established in [16]. Short-time existence and uniqueness of smooth and weak solutions for time-dependent second-order local MFGs were addressed in [27] and [28], respectively. Analysis of stationary first-order local MFGs in 1-dimensional setting is performed in [17]. Problems on graphs are considered in [30]. MFG models with density constraints (hard congestion) and local coupling are addressed in [39] (second-order case) and [11] (first-order case). To our knowledge, existence of smooth solutions for stationary first-order MFGs with global coupling has not been studied before.

One of the main tools of analysis in MFGs theory is the method of a priori estimates. See [23, 8] and references therein for a detailed account on a priori estimates methods in MFGs.

Here, we take a different route. Firstly, using the 1-dimensional structure of the problem, we reduce it to an equation with only m and \overline{H} as unknowns. Indeed, from the second equation in (1.1) we have that

$$u_x(x) + c = \frac{j}{m(x)^{\alpha-1}}, \quad \forall x \in \mathbb{T}, \quad (1.5)$$

where j is some constant that we call *current*. Therefore, (1.1) can be written in an equivalent form

$$\begin{cases} \frac{j^2}{2m^\alpha(x)} + V(x) = \int_{\mathbb{T}} G(x - y) m(y) dy + \overline{H}, & x \in \mathbb{T} \\ m > 0, \int_{\mathbb{T}} m(x) dx = 1. \end{cases} \quad (1.6)$$

Remark 1.1. From here on, we do not differentiate between (1.1) and (1.6). Moreover, we refer to (1.6) as the original problem.

Remark 1.2. Note that c , as a solution parameter, in (1.1) is replaced by j in (1.6). We discuss the relation between c and j in Section 3.

Following [18], [17] we call (1.6) the *current formulation* of (1.1). There are two possibilities: $j \neq 0$ and $j = 0$. We study the simpler case $j = 0$ only in Section 3 and focus on the case $j \neq 0$ afterwards.

Our main observation is that when G is a trigonometric polynomial solutions of (1.6) have a certain structure in terms of unknown Fourier coefficients that satisfy a related equation.

More precisely, for $j \neq 0$ denote by $c_j = (j^2/2)^{\frac{1}{\alpha}}$. Furthermore, for $0 < \alpha \leq 2$ denote by $\phi_\alpha : (0, +\infty) \rightarrow \mathbb{R}$ the antiderivative of $x \mapsto \frac{c_j}{x^\alpha}$; that is,

$$\phi_\alpha(x) = \begin{cases} \frac{c_j \alpha}{\alpha-1} x^{\frac{\alpha-1}{\alpha}}, & \text{if } \alpha \neq 1, \\ c_j \ln x, & \text{if } \alpha = 1. \end{cases}$$

Next, let \mathcal{C} be the set of all points $(a_0, \dots, a_n, b_1, \dots, b_n) \in \mathbb{R}^{2n+1}$ such that

$$a_0 + \sum_{k=1}^n a_k \cos(2\pi kx) + b_k \sin(2\pi kx) - V(x) > 0, \text{ for all } x \in \mathbb{T}. \quad (1.7)$$

Finally, for $(a_0, \dots, a_n, b_1, \dots, b_n) \in \mathcal{C}$ define

$$\begin{aligned} \Phi_\alpha(a_0, a_1, \dots, a_n, b_1, \dots, b_n) = \\ \int_{\mathbb{T}} \phi_\alpha \left(a_0 + \sum_{k=1}^n a_k \cos(2\pi ky) + b_k \sin(2\pi ky) - V(y) \right) dy. \end{aligned} \quad (1.8)$$

Then, we prove the following theorem.

Theorem 1.3. *Suppose that G is a trigonometric polynomial; that is,*

$$G(x) = p_0 + \sum_{k=1}^n p_k \cos(2\pi kx) + \sum_{k=1}^n q_k \sin(2\pi kx) \quad (1.9)$$

for some $n \in \mathbb{N}$ and $p_0, p_1, \dots, p_n, q_1, \dots, q_n \in \mathbb{R}$. Then, if G satisfies (2.1) and (2.2) the system (1.6) has a unique smooth solution.

Moreover, the solution (m, \bar{H}) of (1.6) is given by formulas

$$m(x) = \frac{c_j}{\left(a_0^* + \sum_{k=1}^n a_k^* \cos(2\pi kx) + b_k^* \sin(2\pi kx) - V(x) \right)^{\frac{1}{\alpha}}}, \quad (1.10)$$

and

$$\bar{H} = a_0^* - p_0, \quad (1.11)$$

where $(a_0^*, a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*)$ is the unique solution of the system

$$\begin{cases} \frac{\partial \Phi_\alpha}{\partial a_0} = 1, \\ \frac{\partial \Phi_\alpha}{\partial a_k} = \frac{p_k}{p_k^2 + q_k^2} a_k + \frac{q_k}{p_k^2 + q_k^2} b_k \\ \frac{\partial \Phi_\alpha}{\partial b_k} = \frac{p_k}{p_k^2 + q_k^2} b_k - \frac{q_k}{p_k^2 + q_k^2} a_k, \quad 1 \leq k \leq n, \end{cases} \quad (1.12)$$

where Φ_α is given by (1.8).

Remark 1.4. Assumptions (2.1), (2.2) are natural monotonicity assumptions for the coupling $\int_{\mathbb{T}} G(x-y)m(y)dy$, and we discuss them in Section 2. When G has the form (1.9) these assumptions are equivalent to $p_k \geq 0$, $0 \leq k \leq n$ and $p_0 > 0$, respectively (see Section 4).

Theorem 1.3 reduces the a priori-infinite-dimensional problem (1.6) to a finite dimensional problem (1.12) when the kernel is a trigonometric polynomial. Also, Φ_α is concave, so (1.12) corresponds to finding a root of a monotone mapping which is advantageous from the numerical perspective. This reduction is even more substantial, when the kernel G is a symmetrical trigonometric polynomial; that is, $q_k = 0$ for $1 \leq k \leq n$. In the latter case, (1.12) is equivalent to a concave optimization problem. More precisely, we obtain the following corollary.

Corollary 1.5. *Suppose that G is a symmetrical trigonometric polynomial; that is,*

$$G(x) = \sum_{k=0}^n p_k \cos(2\pi kx) \quad (1.13)$$

for some $n \in \mathbb{N}$ and $p_0, p_1, \dots, p_n \in \mathbb{R}$. Then, if G satisfies (2.1) and (2.2) the system (1.6) has a unique smooth solution.

Moreover, the solution (m, \overline{H}) of (1.6) is given by formulas (1.10) and (1.11) where $(a_0^*, a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*)$ is the unique solution of the optimization problem

$$\max_{(a_0, a_1, \dots, a_n, b_1, \dots, b_n) \in \mathcal{C}} \left(\Phi_\alpha(a_0, a_1, \dots, a_n, b_1, \dots, b_n) - a_0 - \sum_{k=1}^n \frac{1}{2p_k} (a_k^2 + b_k^2) \right). \quad (1.14)$$

Additionally, we find closed form solutions in some special cases.

Theorem 1.6. Assume that $\alpha = 1$ and G, V are first-order trigonometric polynomials; that is

$$G(x) = p_0 + p_1 \cos(2\pi x) + q_1 \sin(2\pi x), \quad V(x) = v_0 + v_1 \cos(2\pi x) + w_1 \sin(2\pi x),$$

where $p_0, p_1, q_1, v_0, v_1, w_1 \in \mathbb{R}$ and $p_0 > 0, p_1 \geq 0$. Then, define $a_0, a_1, b_1, \overline{H}$ as follows:

$$\begin{cases} a_0 &= 2r - 1, \\ \overline{H} &= \frac{j^2(2r-1)}{2} + v_0 - p_0 \\ a_1 &= -\frac{2r(v_1(p_1+j^2r)+w_1q_1)}{(p_1+j^2r)^2+q_1^2}, \\ b_1 &= -\frac{2r(w_1(p_1+j^2r)-v_1q_1)}{(p_1+j^2r)^2+q_1^2}, \end{cases} \quad (1.15)$$

where r is the unique number that satisfies the following equation

$$\left(1 - \frac{1}{r}\right) \left((p_1 + j^2r)^2 + q_1^2\right) = v_1^2 + w_1^2 \quad \text{and} \quad r \geq 1. \quad (1.16)$$

Then the pair $(m(x), \overline{H})$, where

$$m(x) = \frac{1}{a_0 + a_1 \cos(2\pi x) + b_1 \sin(2\pi x)}, \quad (1.17)$$

is the unique solution of (1.6).

Besides the trigonometric-polynomial case we also study (1.6) for general G . In the latter case, we approximate G by trigonometric polynomials and recover the solution of (1.6) as the limit of solutions of approximate problems. More precisely, we prove the following theorem.

Theorem 1.7. Let $G \in C^2(\mathbb{T})$, $V \in C^2(\mathbb{T})$ and G satisfies (2.1), (2.2). Then, there exists a sequence of trigonometric polynomials $\{G_n\}_{n \in \mathbb{N}}$ such that

- i. G_n satisfies (2.1) and (2.2) for all $n \in \mathbb{N}$,
- ii. $\lim_{n \rightarrow \infty} \|G - G_n\|_{C^2(\mathbb{T})} = 0$.

Furthermore, for $n \in \mathbb{N}$ denote by $(m_n, \overline{H}_n) \in C^2(\mathbb{T}) \times \mathbb{R}$ the solution of (1.6) corresponding to G_n (the existence of this solution is guaranteed by Theorem 1.3). Then, there exists $(m, \overline{H}) \in C^2(\mathbb{T}) \times \mathbb{R}$ such that

$$\begin{cases} \lim_{n \rightarrow \infty} \|m - m_n\|_{C^2(\mathbb{T})} = 0, \\ \lim_{n \rightarrow \infty} (\overline{H} - \overline{H}_n) = 0. \end{cases} \quad (1.18)$$

Consequently, (m, \overline{H}) is the unique smooth solution of (1.6) corresponding to G .

In combination with preceding results this previous theorem provides a convenient method for numerical calculations of solutions of (1.6).

We also present a possible way to apply our methods to more general one-dimensional MFG models. We consider the following generalization of (1.1)

$$\begin{cases} H(x, u_x, m) = \mathcal{F}\left(\int_{\mathbb{T}} G(x-y)m(y)dy\right) + \overline{H}, \\ (mH'_p(x, u_x, m))_x = 0, \\ m > 0, \int_{\mathbb{T}} m(x)dx = 1. \end{cases} \quad (1.19)$$

In (1.19), G is a given kernel, $H : \mathbb{T} \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$, $(x, p, m) \mapsto H(x, p, m)$, is a given Hamiltonian, and $\mathcal{F} : X \rightarrow \mathbb{R}$ is a given coupling, where X can be a functional space or \mathbb{R} . We discuss, formally, how our techniques apply to models like (1.19).

The paper is organized as follows. In Section 2 we present the main assumptions and notation. In Section 3 we study (1.6) for the case $j = 0$.

Next, in Section 4 we analyze (1.6) when G is a trigonometric polynomial and prove Theorem 1.3, Corollary 1.5 and Theorem 1.6. In Section 5 we analyze (1.6) for a general G and prove Theorem 1.7.

In Section 6 we present some numerical experiments. Finally, in Section 7 we discuss possible extensions of our results and a future work.

2. ASSUMPTIONS

Throughout the paper we assume that $G \in C^2(\mathbb{T})$, $V \in C^2(\mathbb{T})$. Moreover, we always assume that

$$\int_{\mathbb{T}^2} G(x-y)f(x)f(y)dx dy \geq 0, \quad (2.1)$$

for all $f \in C(\mathbb{T})$ and

$$\int_{\mathbb{T}} G(y)dy > 0. \quad (2.2)$$

Denote by $G[m](x) = \int_{\mathbb{T}} G(x-y)m(y)dy$ the coupling in (1.1). Then, (2.1) is equivalent to the condition

$$\langle G[m_2] - G[m_1], m_2 - m_1 \rangle \geq 0, \text{ for all } m_1, m_2,$$

that is the monotonicity of the coupling $G[m]$ and plays an essential role in our analysis. In general, monotonicity of the coupling is fundamental in the regularity theory for MFGs: system (1.2) degenerates in several directions if the coupling is not monotone. In the view of (1.3) monotonicity means that agents prefer sparsely populated areas. See [13] and [18] for a systematic study of non-monotone MFGs.

Assumption (2.2) is a technical assumption. It is not restrictive since one can always modify the kernel by adding a positive constant.

Furthermore, we assume that

$$0 < \alpha \leq 2.$$

This, also, is a natural assumption for MFGs from the regularity theory perspective. The, now standard, uniqueness proof for MFG systems in [35] is valid only for α in this range. This is a strong indication of degeneracy for α outside of this range (which is observed and discussed in detail in [17]). In fact, our methods also reflect these limitations in a natural way.

3. THE 0-CURRENT CASE

As we have pointed out in the Introduction, (1.1) can be reduced to (1.6) by eliminating u from the second equation. The analysis of (1.6) is completely different for the case $j = 0$ and for the case $j \neq 0$. In fact, the case $j = 0$ is much simpler to analyze. Nevertheless, it is more degenerate. In this section, we discuss the case $j = 0$.

Firstly, we observe that $j = 0$ can occur only when $c = 0$. Recall that in this paper we are concerned only with smooth solutions. Therefore, if (u, m, \overline{H}) is a solution of (1.1) we obtain (1.5) and

$$c = \int_{\mathbb{T}} (u_x(x) + c) dx = j \int_{\mathbb{T}} \frac{dx}{m(x)^{\alpha-1}}.$$

Hence, $j = 0$ if and only if $c = 0$. Furthermore, if $c = j = 0$ (1.1) reduces to

$$\begin{cases} V(x) = \int_{\mathbb{T}} G(x-y)m(y)dy + \overline{H}, & x \in \mathbb{T} \\ m > 0, \int_{\mathbb{T}} m(x)dx = 1. \end{cases} \quad (3.1)$$

At this point, we drop assumptions (2.1) and (2.2) because they are irrelevant. Suppose that V, G, m have following Fourier expansions

$$\begin{aligned} V(x) &= \sum_{k=0}^{\infty} v_k \cos(2\pi kx) + \sum_{k=1}^{\infty} w_k \sin(2\pi kx) \\ G(x) &= \sum_{k=0}^{\infty} p_k \cos(2\pi kx) + \sum_{k=1}^{\infty} q_k \sin(2\pi kx) \\ m(x) &= \sum_{k=0}^{\infty} a_k \cos(2\pi kx) + \sum_{k=1}^{\infty} b_k \sin(2\pi kx) \end{aligned}$$

Then, (3.1) is equivalent to

$$\begin{cases} v_0 &= p_0 a_0 + \overline{H} \\ v_k &= \frac{1}{2}(p_k a_k - q_k b_k) \\ w_k &= \frac{1}{2}(p_k b_k + q_k a_k), & k \geq 1 \\ a_0 &= 1 \\ m &> 0. \end{cases} \quad (3.2)$$

Therefore, we get that

$$\overline{H} = v_0 - p_0, \quad (3.3)$$

and

$$\begin{cases} a_k &= \frac{2(p_k v_k + q_k w_k)}{p_k^2 + q_k^2} \\ b_k &= \frac{2(p_k w_k - q_k v_k)}{p_k^2 + q_k^2}, & k \geq 1. \end{cases}$$

Hence, formally, if $p = 0$ and V, G are given, we obtain that \overline{H} is given by (3.3) and

$$m(x) = 1 + \sum_{k=1}^{\infty} \frac{2}{p_k^2 + q_k^2} ((p_k v_k + q_k w_k) \cos(2\pi kx) + (p_k w_k - q_k v_k) \sin(2\pi kx)) \quad (3.4)$$

$$= 1 + \sum_{k=1}^{\infty} \frac{2}{r_k} (v_k \cos(2\pi kx + \theta_k) + w_k \sin(2\pi kx + \theta_k)),$$

where $r_k e^{i\theta_k} = p_k + iq_k$.

Nevertheless, there are several issues in the previous analysis. Firstly, (3.2) may fail to have solutions or may have infinite number of solutions. If $p_k^2 + q_k^2 = 0$ for some $k \geq 1$ and $v_k^2 + w_k^2 > 0$ then (3.2) and (3.1) do not have solutions. On the other hand if $p_k^2 + q_k^2 = v_k^2 + w_k^2 = 0$ then a_k, b_k can be chosen arbitrarily, so (3.2) and (3.1) may have infinite number of solutions. Thus, if $p_k^2 + q_k^2 = 0$ for some $k \geq 1$ (1.6) degenerates in different ways when $j = 0$.

Furthermore, if $p_k^2 + q_k^2 > 0$ for all $k \geq 1$ then m , at least formally, is given by (3.4). Here we face two potential problems. First, one has to make sense of the formula (3.4). In other words, the series in (3.4) may not be summable in any appropriate sense. Moreover, summability of (3.4) is a delicate issue and strongly depends on the relation between $\{v_k, w_k\}$ and $\{p_k, q_k\}$.

Additionally, even if the series (3.4) converge to a smooth function, we still have the necessary condition $m(x) > 0$ and that might fail depending on V and G . For instance, if V is such that $v_k = w_k = 0$ for all $k \geq 2$, and G is such that $p_k^2 + q_k^2 > 0$ for all $k \geq 1$ we get that

$$m(x) = 1 + \frac{2}{r_1} (v_1 \cos(2\pi x + \theta_1) + w_1 \sin(2\pi x + \theta_1)).$$

Therefore,

$$\min_{x \in \mathbb{T}} m(x) = 1 - \frac{2\sqrt{v_1^2 + w_1^2}}{r_1} = 1 - \frac{2\sqrt{v_1^2 + w_1^2}}{\sqrt{p_1^2 + q_1^2}} > 0$$

if and only if $p_1^2 + q_1^2 > 4(v_1^2 + w_1^2)$. Hence, if the latter is violated (3.1) does not have smooth solutions.

Thus, existence of smooth, positive solutions for (3.1) depends on peculiar properties of V and G . This is quite different in the case $j \neq 0$, where (1.6) obtains smooth, positive solutions under general assumptions on V and G .

4. G IS A TRIGONOMETRIC POLYNOMIAL

From here on, we assume that $j \neq 0$. In this section, our main goal is to prove Theorem 1.3, Corollary 1.5 and Theorem 1.6.

We break the proof of Theorem 1.3 into three steps. Firstly, we show that (1.6) is equivalent to (1.12) - Proposition 4.3. Secondly, we prove that (1.12) has at most one solution - Proposition 4.6. And thirdly, we show that (1.12) has at least one solution - Proposition 4.8.

We use a short-hand notation (\mathbf{x}, \mathbf{y}) for a vector $(x_0, x_1, \dots, x_n, y_1, y_2, \dots, y_n) \in \mathbb{R}^{2n+1}$, where $\mathbf{x} = (x_0, x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$. For every $\mathbf{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ we denote by $\mathbf{x}' = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Here we perform the analysis in terms of Fourier coefficients of G . Hence, we formulate assumptions (2.1), (2.2) in terms of these coefficients.

Lemma 4.1. *For a G given by (1.9) the assumption (2.1) is equivalent to*

$$p_k \geq 0, \quad \text{for } 0 \leq k \leq n. \quad (4.1)$$

Furthermore, the assumption (2.2) is equivalent to

$$p_0 > 0. \quad (4.2)$$

Proof. Let $f \in C(\mathbb{T})$ and

$$c_0 = \int_{\mathbb{T}} f(x) dx,$$

$$c_k = 2 \int_{\mathbb{T}} f(x) \cos(2\pi kx) dx, \quad s_k = 2 \int_{\mathbb{T}} f(x) \sin(2\pi kx) dx, \quad k \geq 1.$$

A straightforward computation yields

$$\int_{\mathbb{T}^2} G(x-y) f(x) f(y) dx dy = p_0 c_0^2 + \frac{1}{4} \sum_{k=1}^n p_k (c_k^2 + s_k^2), \quad \int_{\mathbb{T}} G(x) dx = p_0.$$

The rest of the proof is evident. \square

Remark 4.2. From here on, we assume that (4.1) and (4.2) hold.

Proposition 4.3 (Equivalent formulation). *Let $(m, \overline{H}) \in C(\mathbb{T}) \times \mathbb{R}$ be a solution of (1.6). Then, (m, \overline{H}) is given by formulas (1.10) and (1.11) for some $(a_0, \dots, a_n, b_1, \dots, b_n) \in \mathcal{C}$ that is a solution of (1.12).*

Conversely, if $(a_0, \dots, a_n, b_1, \dots, b_n) \in \mathcal{C}$ is a solution of the system (1.12), then (m, \overline{H}) defined by (1.10) and (1.11) is a solution for (1.6).

Remark 4.4. In our analysis we assume that $p_k^2 + q_k^2 > 0$ for all $1 \leq k \leq n$. This assumption is not restrictive and the results are valid even if $p_k = q_k = 0$ for some $k \geq 1$. Indeed, if $p_k = q_k = 0$, then in (4.4) there will be no terms with $\cos(2\pi kx)$ and $\sin(2\pi kx)$ and in the subsequent analysis we just have to omit the trigonometric monomials $\cos(2\pi kx)$ and $\sin(2\pi kx)$.

Proof of Proposition 4.3. First, we prove the direct implication. Suppose $(m, \overline{H}) \in C(\mathbb{T}) \times \mathbb{R}$ is a solution of (1.6). A straightforward calculation yields

$$\begin{aligned} \int_{\mathbb{T}} G(x-y) m(y) dy &= p_0 u_0 + \frac{1}{2} \sum_{k=1}^n (p_k u_k - q_k v_k) \cos(2\pi kx) \\ &\quad + \frac{1}{2} \sum_{k=1}^n (p_k v_k + q_k u_k) \sin(2\pi kx), \end{aligned}$$

where

$$\begin{aligned} u_0 &= \int_{\mathbb{T}} m(x) dx, \\ u_k &= 2 \int_{\mathbb{T}} m(x) \cos(2\pi kx) dx, \quad v_k = 2 \int_{\mathbb{T}} m(x) \sin(2\pi kx) dx, \quad k \geq 1. \end{aligned} \tag{4.3}$$

Therefore, from (1.6) we obtain

$$\frac{j^2}{2m^\alpha(x)} + V(x) = a_0^* + \sum_{k=1}^n a_k^* \cos(2\pi kx) + b_k^* \sin(2\pi kx), \tag{4.4}$$

which is equivalent to (1.10). The coefficients $\{a_k, b_k\}$ in the previous equation are given by the formulas

$$\begin{aligned} a_0^* &= p_0 u_0 + \overline{H}, \\ a_k^* &= \frac{1}{2} (p_k u_k - q_k v_k), \quad b_k^* = \frac{1}{2} (p_k v_k + q_k u_k), \quad 1 \leq k \leq n. \end{aligned} \tag{4.5}$$

Since $m > 0$ we obtain that $(a_0^*, \dots, a_n^*, b_1^*, \dots, b_n^*) \in \mathcal{C}$.

Furthermore, from (4.3) and (4.5) we obtain that $u_0 = 1$ and (1.11). Next, we plug the expression (1.10) for m in (4.3), and from (4.5) we obtain the following

system

$$\left\{ \begin{aligned} 1 &= \int_{\mathbb{T}} \frac{c_j dy}{\left(a_0^* + \sum_{k=1}^n a_k^* \cos(2\pi ky) + b_k^* \sin(2\pi ky) - V(y) \right)^{\frac{1}{\alpha}}}, \\ a_k^* &= p_k \int_{\mathbb{T}} \frac{c_j \cos(2\pi ky) dy}{\left(a_0^* + \sum_{k=1}^n a_k^* \cos(2\pi ky) + b_k^* \sin(2\pi ky) - V(y) \right)^{\frac{1}{\alpha}}} \\ &\quad - q_k \int_{\mathbb{T}} \frac{c_j \sin(2\pi ky) dy}{\left(a_0^* + \sum_{k=1}^n a_k^* \cos(2\pi ky) + b_k^* \sin(2\pi ky) - V(y) \right)^{\frac{1}{\alpha}}}, \quad 1 \leq k \leq n, \\ b_k^* &= p_k \int_{\mathbb{T}} \frac{c_j \sin(2\pi ky) dy}{\left(a_0^* + \sum_{k=1}^n a_k^* \cos(2\pi ky) + b_k^* \sin(2\pi ky) - V(y) \right)^{\frac{1}{\alpha}}} \\ &\quad + q_k \int_{\mathbb{T}} \frac{c_j \cos(2\pi ky) dy}{\left(a_0^* + \sum_{k=1}^n a_k^* \cos(2\pi ky) + b_k^* \sin(2\pi ky) - V(y) \right)^{\frac{1}{\alpha}}}, \quad 1 \leq k \leq n. \end{aligned} \right. \quad (4.6)$$

Furthermore, note that

$$\begin{aligned} \frac{\partial \Phi_\alpha}{\partial a_0} &= \int_{\mathbb{T}} \frac{c_j dy}{\left(a_0 + \sum_{k=1}^n a_k \cos(2\pi ky) + b_k \sin(2\pi ky) - V(y) \right)^{\frac{1}{\alpha}}} \\ \frac{\partial \Phi_\alpha}{\partial a_k} &= \int_{\mathbb{T}} \frac{c_j \cos(2\pi ky) dy}{\left(a_0 + \sum_{k=1}^n a_k \cos(2\pi ky) + b_k \sin(2\pi ky) - V(y) \right)^{\frac{1}{\alpha}}} \\ \frac{\partial \Phi_\alpha}{\partial b_k} &= \int_{\mathbb{T}} \frac{c_j \sin(2\pi ky) dy}{\left(a_0 + \sum_{k=1}^n a_k \cos(2\pi ky) + b_k \sin(2\pi ky) - V(y) \right)^{\frac{1}{\alpha}}}, \end{aligned} \quad (4.7)$$

for $1 \leq k \leq n$. Therefore, (4.6) can be written as

$$\left\{ \begin{aligned} 1 &= \frac{\partial \Phi_\alpha(\mathbf{a}^*, \mathbf{b}^*)}{\partial a_0}, \\ a_k &= p_k \frac{\partial \Phi_\alpha(\mathbf{a}^*, \mathbf{b}^*)}{\partial a_k} - q_k \frac{\partial \Phi_\alpha(\mathbf{a}^*, \mathbf{b}^*)}{\partial b_k}, \\ b_k &= p_k \frac{\partial \Phi_\alpha(\mathbf{a}^*, \mathbf{b}^*)}{\partial b_k} + q_k \frac{\partial \Phi_\alpha(\mathbf{a}^*, \mathbf{b}^*)}{\partial a_k}, \quad 1 \leq k \leq n, \end{aligned} \right.$$

where $(\mathbf{a}^*, \mathbf{b}^*) = (a_0^*, \dots, a_n^*, b_1^*, \dots, b_n^*)$. But this previous system is equivalent to (1.12).

The proof of the converse implication is the repetition of previous arguments in the reversed order. \square

Next, we study some properties of \mathcal{C} and Φ_α .

Lemma 4.5. *The following statements hold.*

- i. \mathcal{C} is convex and open.
- ii. $\Phi_\alpha \in C^\infty(\mathcal{C})$.
- iii. For all $(\mathbf{a}, \mathbf{b}) \in \mathcal{C}$

$$\begin{aligned} \frac{\partial \Phi_\alpha(\mathbf{a}, \mathbf{b})}{\partial a_l \partial a_r} &= -\frac{c_j}{\alpha} \int_{\mathbb{T}} \frac{\cos(2\pi ly) \cos(2\pi ry) dy}{\left(a_0 + \sum_{k=1}^n a_k \cos(2\pi ky) + b_k \sin(2\pi ky) - V(y) \right)^{1+\frac{1}{\alpha}}} \\ \frac{\partial \Phi_\alpha(\mathbf{a}, \mathbf{b})}{\partial b_l \partial b_r} &= -\frac{c_j}{\alpha} \int_{\mathbb{T}} \frac{\sin(2\pi ly) \sin(2\pi ry) dy}{\left(a_0 + \sum_{k=1}^n a_k \cos(2\pi ky) + b_k \sin(2\pi ky) - V(y) \right)^{1+\frac{1}{\alpha}}} \end{aligned} \quad (4.8)$$

$$\frac{\partial \Phi_\alpha(\mathbf{a}, \mathbf{b})}{\partial a_l \partial b_r} = -\frac{c_j}{\alpha} \int_{\mathbb{T}} \frac{\cos(2\pi l y) \sin(2\pi r y) dy}{\left(a_0 + \sum_{k=1}^n a_k \cos(2\pi k y) + b_k \sin(2\pi k y) - V(y) \right)^{1+\frac{1}{\alpha}}}.$$

iv. Φ_α is strictly concave. Moreover, for all $(\mathbf{a}, \mathbf{b}) \in \mathcal{C}$ and $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^{2n+1}$ we have that

$$\begin{aligned} & (\boldsymbol{\xi}, \boldsymbol{\eta})^T D_{\mathbf{a}, \mathbf{b}}^2 \Phi_\alpha(\mathbf{a}, \mathbf{b})(\boldsymbol{\xi}, \boldsymbol{\eta}) \\ &= -\frac{c_j}{\alpha} \int_{\mathbb{T}} \frac{\left(\eta_0 + \sum_{k=1}^n \xi_k \cos(2\pi k y) + \eta_k \sin(2\pi k y) \right)^2 dy}{\left(a_0 + \sum_{k=1}^n a_k \cos(2\pi k y) + b_k \sin(2\pi k y) - V(y) \right)^{1+\frac{1}{\alpha}}} \\ &\leq 0, \end{aligned} \tag{4.9}$$

with equality if and only if $(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathbf{0}$.

v. $-\nabla \Phi_\alpha$ is strictly monotone; that is, for all $(\mathbf{c}_1, \mathbf{d}_1), (\mathbf{c}_2, \mathbf{d}_2) \in \mathcal{C}$

$$\langle \nabla \Phi_\alpha(\mathbf{c}_2, \mathbf{d}_2) - \nabla \Phi_\alpha(\mathbf{c}_1, \mathbf{d}_1), (\mathbf{c}_2 - \mathbf{c}_1, \mathbf{d}_2 - \mathbf{d}_1) \rangle \leq 0, \tag{4.10}$$

with equality if and only if $(\mathbf{c}_1, \mathbf{d}_1) = (\mathbf{c}_2, \mathbf{d}_2)$.

Proof. i. This statement is evident.

ii. This statement is evident.

iii. We obtain (4.8) by a straightforward calculation.

iv. Equation (4.9) follows from (4.8) by an algebraic manipulation. Moreover, the equality in (4.9) holds if and only if

$$\eta_0 + \sum_{k=1}^n \xi_k \cos(2\pi k y) + \eta_k \sin(2\pi k y) = 0, \text{ for all } y \in \mathbb{T},$$

which implies $(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathbf{0}$.

v. For $t \in [0, 1]$ denote by $(\mathbf{c}(t), \mathbf{d}(t)) = (1-t)(\mathbf{c}_1, \mathbf{d}_1) + t(\mathbf{c}_2, \mathbf{d}_2)$. Since \mathcal{C} is convex, we have that $(\mathbf{c}(t), \mathbf{d}(t)) \in \mathcal{C}, t \in [0, 1]$. Furthermore, denote by $f(t) = \Phi_\alpha(\mathbf{c}(t), \mathbf{d}(t))$. We have that $f \in C^\infty([0, 1])$ because $\Phi_\alpha \in C^\infty(\mathcal{C})$. Moreover, by (4.9) we have that

$$f''(t) = (\mathbf{c}_2 - \mathbf{c}_1, \mathbf{d}_2 - \mathbf{d}_1)^T D_{\mathbf{a}, \mathbf{b}}^2 \Phi_\alpha(\mathbf{c}(t), \mathbf{d}(t))(\mathbf{c}_2 - \mathbf{c}_1, \mathbf{d}_2 - \mathbf{d}_1) < 0$$

for all $t \in [0, 1]$, unless $(\mathbf{c}_1, \mathbf{d}_1) = (\mathbf{c}_2, \mathbf{d}_2)$. Hence,

$$f'(1) - f'(0) \leq 0,$$

with equality if and only if $(\mathbf{c}_1, \mathbf{d}_1) = (\mathbf{c}_2, \mathbf{d}_2)$. We complete the proof by noting that

$$\langle \nabla \Phi_\alpha(\mathbf{c}_2, \mathbf{d}_2) - \nabla \Phi_\alpha(\mathbf{c}_1, \mathbf{d}_1), (\mathbf{c}_2 - \mathbf{c}_1, \mathbf{d}_2 - \mathbf{d}_1) \rangle = f'(1) - f'(0).$$

□

Proposition 4.6 (Uniqueness.). *If $(\mathbf{c}_1, \mathbf{d}_1), (\mathbf{c}_2, \mathbf{d}_2) \in \mathcal{C}$ are solutions of (1.12), then $(\mathbf{c}_1, \mathbf{d}_1) = (\mathbf{c}_2, \mathbf{d}_2)$.*

Proof. Let $(\mathbf{c}_i, \mathbf{d}_i) = (c_{i0}, c_{i1}, \dots, c_{in}, d_{i1}, \dots, d_{in})$ for $i = 1, 2$. Then, we have that

$$\begin{cases} \frac{\partial \Phi_\alpha(\mathbf{c}_i, \mathbf{d}_i)}{\partial a_0} &= 1, \\ \frac{\partial \Phi_\alpha(\mathbf{c}_i, \mathbf{d}_i)}{\partial a_k} &= \frac{p_k}{p_k^2 + q_k^2} c_{ik} + \frac{q_k}{p_k^2 + q_k^2} d_{ik} \\ \frac{\partial \Phi_\alpha(\mathbf{c}_i, \mathbf{d}_i)}{\partial b_k} &= \frac{p_k}{p_k^2 + q_k^2} d_{ik} - \frac{q_k}{p_k^2 + q_k^2} c_{ik}, \quad 1 \leq k \leq n, \end{cases}$$

for $i = 1, 2$. Hence,

$$\begin{aligned} S &:= \langle \nabla \Phi_\alpha(\mathbf{c}_2, \mathbf{d}_2) - \nabla \Phi_\alpha(\mathbf{c}_1, \mathbf{d}_1), (\mathbf{c}_2 - \mathbf{c}_1, \mathbf{d}_2 - \mathbf{d}_1) \rangle \\ &= \sum_{k=1}^n \frac{p_k}{p_k^2 + q_k^2} (c_{2k} - c_{1k})^2 + \frac{p_k}{p_k^2 + q_k^2} (d_{2k} - d_{1k})^2 \geq 0. \end{aligned}$$

On the other hand, from (4.10) we have that $S \leq 0$, so $S = 0$ and $(\mathbf{c}_1, \mathbf{d}_1) = (\mathbf{c}_2, \mathbf{d}_2)$. \square

Lemma 4.7. *For every $(\mathbf{a}', \mathbf{b}) \in \mathbb{R}^{2n}$ there exists a unique $a_0 = \omega(\mathbf{a}', \mathbf{b}) \in \mathbb{R}$ such that*

$$\frac{\partial \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0} = 1. \quad (4.11)$$

Furthermore, $\omega \in C^\infty(\mathbb{R}^{2n})$.

Proof. Fix a point $(\mathbf{a}', \mathbf{b}) \in \mathbb{R}^{2n}$. Denote by

$$\begin{aligned} l(\mathbf{a}', \mathbf{b}) &= - \inf_{x \in \mathbb{T}} \left(\sum_{k=1}^n (a_k \cos(2\pi kx) + b_k \sin(2\pi kx)) - V(x) \right) \\ &= - \sum_{k=1}^n (a_k \cos(2\pi kx_0) + b_k \sin(2\pi kx_0)) + V(x_0), \end{aligned}$$

where $x_0 \in \mathbb{T}$. Firstly, we show that

$$\lim_{a_0 \rightarrow l(\mathbf{a}', \mathbf{b})} \frac{\partial \Phi_\alpha(a_0, \mathbf{a}', \mathbf{b})}{\partial a_0} = \infty.$$

Denote by

$$f(x) = a_0 + \sum_{k=1}^n (a_k \cos(2\pi kx) + b_k \sin(2\pi kx)) - V(x).$$

Then we have that $x_0 \in \operatorname{argmin} f$. Hence, $f'(x_0) = 0$, and

$$f(x) \leq f(x_0) + C(x - x_0)^2 = a_0 - l(\mathbf{a}', \mathbf{b}) + C(x - x_0)^2,$$

where $C = C(\mathbf{a}', \mathbf{b}) = \frac{1}{2} \sup_{x \in \mathbb{T}} |f''(x)|$. Therefore, we have that

$$\begin{aligned} \frac{\partial \Phi_\alpha(a_0, \mathbf{a}', \mathbf{b})}{\partial a_0} &= c_j \int_{\mathbb{T}} \frac{dx}{f(x)^{1/\alpha}} = c_j \int_{x_0-1/2}^{x_0+1/2} \frac{dx}{f(x)^{1/\alpha}} \\ &\geq c_j \int_{x_0-1/2}^{x_0+1/2} \frac{dx}{(a_0 - l(\mathbf{a}', \mathbf{b}) + C(x - x_0)^2)^{1/\alpha}} \\ &= c_j \int_{-1/2}^{1/2} \frac{dx}{(a_0 - l(\mathbf{a}', \mathbf{b}) + Cx^2)^{1/\alpha}} \\ &= c_j (a_0 - l(\mathbf{a}', \mathbf{b}))^{\frac{1}{2} - \frac{1}{\alpha}} \int_{-\frac{1}{2\sqrt{(a_0 - l(\mathbf{a}', \mathbf{b}))}}^{\frac{1}{2\sqrt{(a_0 - l(\mathbf{a}', \mathbf{b}))}}}} \frac{dx}{(1 + Cx^2)^{1/\alpha}}. \end{aligned}$$

For $0 < \alpha \leq 2$ we have that

$$\lim_{\delta \rightarrow 0} \delta^{\frac{1}{2} - \frac{1}{\alpha}} \int_{-\frac{1}{2\sqrt{\delta}}}^{\frac{1}{2\sqrt{\delta}}} \frac{dx}{(1 + Cx^2)^{1/\alpha}} = \infty,$$

so

$$\lim_{a_0 \rightarrow l(\mathbf{a}', \mathbf{b})} \frac{\partial \Phi_\alpha(a_0, \mathbf{a}', \mathbf{b})}{\partial a_0} = \infty.$$

Finally, the mapping $a_0 \mapsto \frac{\partial \Phi_\alpha(a_0, \mathbf{a}', \mathbf{b})}{\partial a_0}$ is decreasing and

$$\lim_{a_0 \rightarrow \infty} \frac{\partial \Phi_\alpha(a_0, \mathbf{a}', \mathbf{b})}{\partial a_0} = 0,$$

so there exists unique $a_0 = \omega(\mathbf{a}', \mathbf{b})$ such that (4.11) holds. Regularity of ω follows from the implicit function theorem. \square

Proposition 4.8 (Existence.). *Let $F : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be the following function:*

$$\begin{aligned} F(\mathbf{a}', \mathbf{b}) &= \frac{1}{2} \sum_{k=1}^n \left(\frac{\partial \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k} - \frac{p_k}{p_k^2 + q_k^2} a_k - \frac{q_k}{p_k^2 + q_k^2} b_k \right)^2 \\ &\quad + \frac{1}{2} \sum_{k=1}^n \left(\frac{\partial \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k} - \frac{p_k}{p_k^2 + q_k^2} b_k + \frac{q_k}{p_k^2 + q_k^2} a_k \right)^2, \end{aligned} \quad (4.12)$$

where $(\mathbf{a}', \mathbf{b}) = (a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)$ and ω is the function from Lemma 4.7. Then, F is bounded by below and coercive. Consequently, the minimization problem

$$\min_{(\mathbf{a}', \mathbf{b}) \in \mathbb{R}^{2n}} F(\mathbf{a}', \mathbf{b}) \quad (4.13)$$

admits at least one solution.

Moreover, if $(\mathbf{a}', \mathbf{b})$ is a critical point for F , then $(\mathbf{a}, \mathbf{b}) = (\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})$ is a solution of (1.12). Therefore, (1.12) admits at least one solution.

Proof. Firstly, we show that F from (4.12) is coercive and bounded by below. Evidently, $F \geq 0$. Next, from (4.7) we have that for all $(\mathbf{a}', \mathbf{b}) \in \mathbb{R}^{2n}$

$$\left| \frac{\partial \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k} \right|, \left| \frac{\partial \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k} \right| \leq \frac{\partial \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0} = 1$$

for $1 \leq k \leq n$. Furthermore, we use the elementary inequality

$$(x - y)^2 \geq \frac{x^2}{2} - y^2 \geq \frac{x^2}{2} - 1, \text{ for } x \in \mathbb{R}, |y| \leq 1,$$

and obtain

$$\begin{aligned} F(\mathbf{a}', \mathbf{b}) &\geq \frac{1}{2} \sum_{k=1}^n \left[\frac{1}{2} \left(\frac{p_k}{p_k^2 + q_k^2} a_k + \frac{q_k}{p_k^2 + q_k^2} b_k \right)^2 - 1 \right] \\ &\quad + \frac{1}{2} \sum_{k=1}^n \left[\frac{1}{2} \left(\frac{p_k}{p_k^2 + q_k^2} b_k - \frac{q_k}{p_k^2 + q_k^2} a_k \right)^2 - 1 \right] \\ &= \frac{1}{4} \sum_{k=1}^n \frac{a_k^2 + b_k^2}{p_k^2 + q_k^2} - n, \end{aligned}$$

for all $(\mathbf{a}', \mathbf{b}) \in \mathbb{R}^{2n}$. Therefore, F is coercive.

Now, we prove that for every critical point $(\mathbf{a}', \mathbf{b})$ of F the point $(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})$ is a solution of (1.12). For $1 \leq k \leq n$ denote by

$$\begin{aligned}\xi_k &= \frac{\partial \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k} - \frac{p_k}{p_k^2 + q_k^2} a_k - \frac{q_k}{p_k^2 + q_k^2} b_k \\ \eta_k &= \frac{\partial \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k} - \frac{p_k}{p_k^2 + q_k^2} b_k + \frac{q_k}{p_k^2 + q_k^2} a_k.\end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial F(\mathbf{a}', \mathbf{b})}{\partial a_l} &= \sum_{k=1}^n \xi_k \left(\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial a_l} - \frac{p_k \delta_{kl}}{p_k^2 + q_k^2} + \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial a_0} \cdot \frac{\partial \omega}{\partial a_l} \right) \\ &\quad + \sum_{k=1}^n \eta_k \left(\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial a_l} + \frac{q_k \delta_{kl}}{p_k^2 + q_k^2} + \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial a_0} \cdot \frac{\partial \omega}{\partial a_l} \right), \\ \frac{\partial F(\mathbf{a}', \mathbf{b})}{\partial b_l} &= \sum_{k=1}^n \xi_k \left(\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial b_l} - \frac{q_k \delta_{kl}}{p_k^2 + q_k^2} + \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial a_0} \cdot \frac{\partial \omega}{\partial b_l} \right) \\ &\quad + \sum_{k=1}^n \eta_k \left(\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial b_l} - \frac{p_k \delta_{kl}}{p_k^2 + q_k^2} + \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial a_0} \cdot \frac{\partial \omega}{\partial b_l} \right),\end{aligned}$$

for $1 \leq l \leq n$. Next, by differentiating (4.11) we obtain

$$\frac{\partial \omega(\mathbf{a}', \mathbf{b})}{\partial a_l} = - \frac{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0 \partial a_l}}{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2}}, \quad \frac{\partial \omega(\mathbf{a}', \mathbf{b})}{\partial b_l} = - \frac{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0 \partial b_l}}{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2}},$$

for $1 \leq l \leq n$.

Now, suppose $(\mathbf{a}', \mathbf{b}) \in \mathbb{R}^n$ is a minimizer of (4.13). Then, we have that

$$\begin{aligned}0 &= \sum_{l=1}^n \xi_l \frac{\partial F(\mathbf{a}', \mathbf{b})}{\partial a_l} + \sum_{l=1}^n \eta_l \frac{\partial F(\mathbf{a}', \mathbf{b})}{\partial b_l} \\ &= \sum_{l,k} \xi_l \xi_k \left(\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial a_l} - \frac{p_k \delta_{kl}}{p_k^2 + q_k^2} + \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial a_0} \cdot \frac{\partial \omega}{\partial a_l} \right) \\ &\quad + \sum_{l,k} \xi_l \eta_k \left(\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial a_l} + \frac{q_k \delta_{kl}}{p_k^2 + q_k^2} + \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial a_0} \cdot \frac{\partial \omega}{\partial a_l} \right) \\ &\quad + \sum_{l,k} \eta_l \xi_k \left(\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial b_l} - \frac{q_k \delta_{kl}}{p_k^2 + q_k^2} + \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial a_0} \cdot \frac{\partial \omega}{\partial b_l} \right) \\ &\quad + \sum_{l,k} \eta_l \eta_k \left(\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial b_l} - \frac{p_k \delta_{kl}}{p_k^2 + q_k^2} + \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial a_0} \cdot \frac{\partial \omega}{\partial b_l} \right) \\ &= (\boldsymbol{\xi}', \boldsymbol{\eta})^T D_{\mathbf{a}', \mathbf{b}}^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b}) (\boldsymbol{\xi}', \boldsymbol{\eta}) - \sum_{k=1}^n \frac{p_k}{p_k^2 + q_k^2} (\xi_k^2 + \eta_k^2) \\ &\quad + \sum_{l,k} \left(\xi_l \xi_k \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial a_0} \cdot \frac{\partial \omega}{\partial a_l} + \xi_l \eta_k \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial a_0} \cdot \frac{\partial \omega}{\partial a_l} \right) \\ &\quad + \sum_{l,k} \left(\eta_l \xi_k \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial a_0} \cdot \frac{\partial \omega}{\partial b_l} + \eta_l \eta_k \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial a_0} \cdot \frac{\partial \omega}{\partial b_l} \right) \\ &= (\boldsymbol{\xi}', \boldsymbol{\eta})^T D_{\mathbf{a}', \mathbf{b}}^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b}) (\boldsymbol{\xi}', \boldsymbol{\eta}) - \sum_{k=1}^n \frac{p_k}{p_k^2 + q_k^2} (\xi_k^2 + \eta_k^2)\end{aligned} \tag{4.14}$$

$$\begin{aligned}
& + \left(\sum_{l=1}^n \xi_l \frac{\partial \omega}{\partial a_l} + \eta_l \frac{\partial \omega}{\partial b_l} \right) \left(\sum_{k=1}^n \xi_k \frac{\partial \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_k \partial a_0} + \eta_k \frac{\partial \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial b_k \partial a_0} \right) \\
& = (\boldsymbol{\xi}', \boldsymbol{\eta})^T D_{\mathbf{a}', \mathbf{b}}^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})(\boldsymbol{\xi}', \boldsymbol{\eta}) - \sum_{k=1}^n \frac{p_k}{p_k^2 + q_k^2} (\xi_k^2 + \eta_k^2) \\
& \quad - \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2} \left(\sum_{l=1}^n \xi_l \frac{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0 \partial a_l}}{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2}} + \eta_l \frac{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0 \partial b_l}}{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2}} \right)^2 \\
& = (\boldsymbol{\xi}', \boldsymbol{\eta})^T D_{\mathbf{a}', \mathbf{b}}^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})(\boldsymbol{\xi}', \boldsymbol{\eta}) - \sum_{k=1}^n \frac{p_k}{p_k^2 + q_k^2} (\xi_k^2 + \eta_k^2) \\
& \quad - \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2} \xi_0^2,
\end{aligned}$$

where $\boldsymbol{\xi}' = (\xi_1, \dots, \xi_n)$, $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$, and

$$\xi_0 = - \sum_{l=1}^n \left(\xi_l \frac{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0 \partial a_l}}{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2}} + \eta_l \frac{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0 \partial b_l}}{\frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2}} \right).$$

Furthermore, we have that

$$\begin{aligned}
& \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2} \xi_0^2 + 2 \sum_{l=1}^n \left(\xi_0 \xi_l \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0 \partial a_l} + \xi_0 \eta_l \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0 \partial b_l} \right) \\
& = \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2} \xi_0^2 - 2 \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2} \xi_0^2 \\
& = - \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2} \xi_0^2.
\end{aligned}$$

Hence, from (4.14) we obtain that

$$\begin{aligned}
0 & = (\boldsymbol{\xi}', \boldsymbol{\eta})^T D_{\mathbf{a}', \mathbf{b}}^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})(\boldsymbol{\xi}', \boldsymbol{\eta}) - \sum_{k=1}^n \frac{p_k}{p_k^2 + q_k^2} (\xi_k^2 + \eta_k^2) \\
& \quad - \frac{\partial^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})}{\partial a_0^2} \xi_0^2 \\
& = (\boldsymbol{\xi}, \boldsymbol{\eta})^T D_{\mathbf{a}, \mathbf{b}}^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})(\boldsymbol{\xi}, \boldsymbol{\eta}) - \sum_{k=1}^n \frac{p_k}{p_k^2 + q_k^2} (\xi_k^2 + \eta_k^2) \\
& \leq (\boldsymbol{\xi}, \boldsymbol{\eta})^T D_{\mathbf{a}, \mathbf{b}}^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})(\boldsymbol{\xi}, \boldsymbol{\eta}),
\end{aligned}$$

where $\boldsymbol{\xi} = (\xi_0, \boldsymbol{\xi}')$. On the other hand, from (4.9) we have that

$$(\boldsymbol{\xi}, \boldsymbol{\eta})^T D_{\mathbf{a}, \mathbf{b}}^2 \Phi_\alpha(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})(\boldsymbol{\xi}, \boldsymbol{\eta}) \leq 0.$$

Therefore, there is an equality in the previous equation and from (4.9) we obtain that $(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathbf{0}$ or that, equivalently, $\xi_k = \eta_k = 0$ for $1 \leq k \leq n$. The latter precisely means that $(\omega(\mathbf{a}', \mathbf{b}), \mathbf{a}', \mathbf{b})$ is a solution of (1.12). \square

Now we are in the position to prove Theorem 1.3.

Proof of the Theorem 1.3. We have that (1.6) is equivalent to (1.12) by Proposition 4.3. Furthermore, (1.12) admits a solution $(\mathbf{a}^*, \mathbf{b}^*)$ by Theorem 4.8. Moreover, $(\mathbf{a}^*, \mathbf{b}^*)$ is unique by Proposition 4.6. Hence, (1.6) admits a unique solution given by (1.10) and (1.11). \square

Next, we prove Corollary 1.5.

Proof of the Corollary 1.5. By Theorem 1.3 we have that (1.6) obtains unique solution, (m, \overline{H}) , given by (1.10) and (1.11), where $(\mathbf{a}^*, \mathbf{b}^*) = (a_0^*, a_1^*, \dots, a_n^*, b_1^* \dots, b_n^*)$ is the unique solution of (1.12). Since G has the form (1.13) we have that $q_k = 0$ for $1 \leq k \leq n$. Therefore, (1.12) can be written as

$$\nabla_{\mathbf{a}, \mathbf{b}} \left(\Phi_\alpha(\mathbf{a}, \mathbf{b}) - a_0 - \sum_{k=1}^n \frac{1}{2p_k} (a_k^2 + b_k^2) \right) \Big|_{(\mathbf{a}, \mathbf{b}) = (\mathbf{a}^*, \mathbf{b}^*)} = 0.$$

Furthermore, by Lemma 4.5 Φ_α is strictly concave on \mathcal{C} (see (1.7) for the definition of \mathcal{C}), so the function

$$(\mathbf{a}, \mathbf{b}) \mapsto \Phi_\alpha(\mathbf{a}, \mathbf{b}) - a_0 - \sum_{k=1}^n \frac{1}{2p_k} (a_k^2 + b_k^2)$$

is also strictly concave on \mathcal{C} . Hence, $(\mathbf{a}^*, \mathbf{b}^*)$ is the unique maximum of (1.14). \square

Finally, we prove Theorem 1.6.

Proof of the Theorem 1.6. Firstly, note that if m is a solution of (1.6) with $\alpha = 1$, then m must necessarily have the form (1.17). Consequently, (1.17) leads to

$$\begin{aligned} & \frac{j^2}{2} (a_0 + a_1 \cos(2\pi x) + b_1 \sin(2\pi x)) + v_0 + v_1 \cos(2\pi x) + w_1 \sin(2\pi x) \quad (4.15) \\ &= p_0 \int_{\mathbb{T}} \frac{dy}{a_0 + a_1 \cos(2\pi y) + b_1 \sin(2\pi y)} + p_1 \int_{\mathbb{T}} \frac{\cos(2\pi(x-y))dy}{a_0 + a_1 \cos(2\pi y) + b_1 \sin(2\pi y)} \\ &+ q_1 \int_{\mathbb{T}} \frac{\sin(2\pi(x-y))dy}{a_0 + a_1 \cos(2\pi y) + b_1 \sin(2\pi y)}. \end{aligned}$$

A direct calculation yields the following identities

$$\begin{aligned} & \int_{\mathbb{T}} \frac{dy}{a_0 + a_1 \cos(2\pi y) + b_1 \sin(2\pi y)} = \frac{1}{\sqrt{a_0^2 - a_1^2 - b_1^2}} \quad (4.16) \\ & \int_{\mathbb{T}} \frac{\cos(2\pi y)dy}{a_0 + a_1 \cos(2\pi y) + b_1 \sin(2\pi y)} = \frac{a_1(\sqrt{a_0^2 - a_1^2 - b_1^2} - a_0)}{(a_1^2 + b_1^2)\sqrt{a_0^2 - a_1^2 - b_1^2}} \\ & \int_{\mathbb{T}} \frac{\sin(2\pi y)dy}{a_0 + a_1 \cos(2\pi y) + b_1 \sin(2\pi y)} = \frac{b_1(\sqrt{a_0^2 - a_1^2 - b_1^2} - a_0)}{(a_1^2 + b_1^2)\sqrt{a_0^2 - a_1^2 - b_1^2}} \end{aligned}$$

Using (4.16) in (4.15) and taking into the account that $\int_{\mathbb{T}} m(x)dx = 1$, we obtain

$$\begin{cases} \frac{1}{\sqrt{a_0^2 - a_1^2 - b_1^2}} &= 1 \\ \frac{j^2}{2} a_0 + v_0 &= \frac{p_0}{\sqrt{a_0^2 - a_1^2 - b_1^2}} + \overline{H} \\ \frac{j^2}{2} a_1 + v_1 &= \frac{(\sqrt{a_0^2 - a_1^2 - b_1^2} - a_0)}{(a_1^2 + b_1^2)\sqrt{a_0^2 - a_1^2 - b_1^2}} (p_1 a_1 - q_1 b_1) \\ \frac{j^2}{2} b_1 + w_1 &= \frac{(\sqrt{a_0^2 - a_1^2 - b_1^2} - a_0)}{(a_1^2 + b_1^2)\sqrt{a_0^2 - a_1^2 - b_1^2}} (p_1 b_1 + q_1 a_1). \end{cases}$$

which can be equivalently written as

$$\begin{cases} \overline{H} &= \frac{j^2}{2}a_0 + v_0 - p_0 \\ \frac{j^2}{2}a_1 + v_1 &= \frac{(1-a_0)}{(a_0^2-1)}(p_1a_1 - q_1b_1) \\ \frac{j^2}{2}b_1 + w_1 &= \frac{(1-a_0)}{(a_0^2-1)}(p_1b_1 + q_1a_1) \\ a_0^2 - a_1^2 - b_1^2 &= 1. \end{cases}$$

We eliminate a_1 and b_1 in the second and third equations and find a_0 from the fourth equation. It is algebraically more appealing to put $a_0 = 2r - 1$. Then, a straightforward calculation yields (1.16).

Since $m > 0$ we have that $a_0 > 0$. Moreover, from the fourth equation in the previous system we have that $a_0^2 \geq 1$, so $a_0 \geq 1$ or $r \geq 1$. Note that the left-hand-side of (1.16) is increasing function in r for $r \geq 1$, and it is equal to 0 at $r = 1$ and to ∞ at $r = \infty$. Therefore, for arbitrary choices of v_1, w_1 there is a unique $r \geq 1$ such that (1.16) holds. This is coherent with the fact that (1.6) obtains a unique smooth solution. Moreover, (1.16) is a cubic equation. Hence, formulas in (1.15) are explicit. \square

5. G IS A GENERAL KERNEL

In this section we prove Theorem 1.7. We divide the proof into two steps. First, we prove that solutions of (1.6) are stable under C^2 perturbation of the kernel. Second, we show that arbitrary C^2 kernel can be approximated by suitable trigonometric polynomials.

Proof of the Theorem 1.7. The uniqueness of the solution for (1.6) follows from the uniqueness of the solution of (1.1) (See [38]).

Part 1. Stability. Suppose that $\{G_n\}_{n \in \mathbb{N}} \subset C^2(\mathbb{T})$ are such that

$$\lim_{n \rightarrow \infty} \|G_n - G\|_{C^2(\mathbb{T})} = 0.$$

Moreover, assume that for each $n \geq 1$ (1.6) has a solution, $(m_n, \overline{H}_n) \in C^2(\mathbb{T}) \times \mathbb{R}$, corresponding to the kernel G_n . We aim to prove that there exists $(m, \overline{H}) \in C^2(\mathbb{T}) \times \mathbb{R}$ such that (1.18) holds and (m, \overline{H}) is the solution of (1.6) corresponding to the kernel G .

Remark 5.1. Note that in this part of the proof we do not assume that $\{G_n\}$ are trigonometric polynomials and that they satisfy (2.1), (2.2). We need these assumptions in the second part of the proof to guarantee the existence of solutions (m_n, \overline{H}_n) .

We are going to show that families

$$\{m_n\}_{n \in \mathbb{N}}, \quad \left\{ \frac{1}{m_n^\alpha} \right\}_{n \in \mathbb{N}}$$

are uniformly bounded and equicontinuous. Denote by

$$m_n(x) = \frac{(j^2/2)^{1/\alpha}}{f_n(x)^{1/\alpha}},$$

where

$$f_n(x) = \int_{\mathbb{T}} G_n(x-y)m_n(y)dy + \overline{H}_n - V(x), \quad x \in \mathbb{T}.$$

We have that

$$\begin{aligned} |f_n''(x)| &= \left| \int_{\mathbb{T}} G_n''(x-y)m_n(y)dy - V''(x) \right| \leq \sup_{\mathbb{T}} |G_n''| \int_{\mathbb{T}} m_n(y)dy + \sup_{\mathbb{T}} |V''| \\ &\leq \sup_n \sup_{\mathbb{T}} |G_n''| + \sup_{\mathbb{T}} |V''| =: 2C, \quad x \in \mathbb{T}. \end{aligned}$$

Next, denote by $\sigma_n := \min_{\mathbb{T}} f_n = f_n(x_n)$, for some $x_n \in \mathbb{T}$. Then, we have that $f'(x_n) = 0$, and

$$f_n(x) \leq \sigma_n + C(x - x_n)^2, \quad x \in \mathbb{T}.$$

Therefore,

$$\begin{aligned} 1 &= \int_{\mathbb{T}} m_n(x)dx = (j^2/2)^{1/\alpha} \int_{\mathbb{T}} \frac{dx}{f_n(x)^{1/\alpha}} = (j^2/2)^{1/\alpha} \int_{x_n-1/2}^{x_n+1/2} \frac{dx}{f_n(x)^{1/\alpha}} \\ &\geq (j^2/2)^{1/\alpha} \int_{x_n-1/2}^{x_n+1/2} \frac{dx}{(\sigma_n + C(x - x_n)^2)^{1/\alpha}} = (j^2/2)^{1/\alpha} \int_{-1/2}^{1/2} \frac{dx}{(\sigma_n + Cx^2)^{1/\alpha}} \\ &= (j^2/2)^{1/\alpha} \sigma_n^{\frac{1}{2}-\frac{1}{\alpha}} \int_{-\frac{1}{2\sqrt{\sigma_n}}}^{\frac{1}{2\sqrt{\sigma_n}}} \frac{dx}{(1 + Cx^2)^{1/\alpha}}. \end{aligned}$$

Furthermore, for $0 < \alpha \leq 2$ we have that

$$\lim_{\delta \rightarrow 0} \delta^{\frac{1}{2}-\frac{1}{\alpha}} \int_{-\frac{1}{2\sqrt{\delta}}}^{\frac{1}{2\sqrt{\delta}}} \frac{dx}{(1 + Cx^2)^{1/\alpha}} = \infty.$$

Therefore, $\sigma_n \geq \delta_0 > 0$, or

$$m_n(x) = \frac{(j^2/2)^{1/\alpha}}{f_n(x)^{1/\alpha}} \leq \frac{(j^2/2)^{1/\alpha}}{\sigma_n^{1/\alpha}} \leq \frac{(j^2/2)^{1/\alpha}}{\delta_0^{1/\alpha}} =: C_1, \quad x \in \mathbb{T}, \quad (5.1)$$

for $n \geq 1$. Furthermore, denote by $m_n(z_n) = \max_{\mathbb{T}} m_n$. Then, we have that

$$m_n(z_n) \geq \int_{\mathbb{T}} m_n(x)dx = 1. \quad (5.2)$$

Furthermore, for every $x, z \in \mathbb{T}$ we have that

$$\begin{aligned} \left| \frac{j^2}{2m_n^\alpha(x)} - \frac{j^2}{2m_n^\alpha(z)} \right| &= \left| \int_{\mathbb{T}} (G_n(x-y) - G_n(z-y))m_n(y)dy - V(x) + V(z) \right| \quad (5.3) \\ &\leq \left(\sup_{\mathbb{T}} |G_n'| + \sup_{\mathbb{T}} |V'| \right) |x - z| \\ &\leq \left(\sup_{n, \mathbb{T}} |G_n'| + \sup_{\mathbb{T}} |V'| \right) |x - z|. \end{aligned}$$

Firstly, if we plug in $z = z_n$ in (5.3) and use (5.2), we get that

$$\frac{1}{m_n^\alpha(x)} \leq C_2, \quad x \in \mathbb{T},$$

for all $n \geq 1$. Secondly, (5.3) yields that the family

$$\left\{ \frac{1}{m_n^\alpha} \right\}_{n \in \mathbb{N}}$$

is uniformly Lipschitz which in turn yields (in combination with (5.1)) that the family $\{m_n^\alpha\}_{n \in \mathbb{N}}$ is also uniformly Lipschitz.

Since families (1.18) are uniformly bounded, we get that $\{\overline{H}_n\}_{n \in \mathbb{N}}$ is a bounded sequence. Then, we can assume that there exists $(m, \overline{H}) \in C(\mathbb{T}) \times \mathbb{R}$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|m_n - m\|_{C(\mathbb{T})} &= 0, \\ \lim_{n \rightarrow \infty} \left\| \frac{1}{m_n^\alpha} - \frac{1}{m^\alpha} \right\|_{C(\mathbb{T})} &= 0, \\ \lim_{n \rightarrow \infty} (\overline{H}_n - \overline{H}) &= 0, \end{aligned}$$

through a subsequence. Moreover, we obtain (1.18) through the same subsequence.

From the previous equations, we obtain that (m, \overline{H}) solves (1.6) for the kernel G . Next, (1.6) must have a unique solution because it is equivalent to (1.1) that can have at most one solution (see [38]). Hence, the limit, (m, \overline{H}) , is the same for all subsequences. Therefore, (1.18) is valid through the whole sequence.

Part 2. Approximation. Suppose $G \in C^2(\mathbb{T})$ satisfies (2.1) and (2.2) are satisfied. We formally expand G in Fourier series

$$G(x) = p_0 + \sum_{k=1}^{\infty} p_k \cos(2\pi kx) + q_k \sin(2\pi kx), \quad x \in \mathbb{T}.$$

Denote by

$$S_n(x) = p_0 + \sum_{k=1}^n p_k \cos(2\pi kx) + q_k \sin(2\pi kx), \quad x \in \mathbb{T}, \quad n \geq 1,$$

and $S_0(x) = p_0$ the truncated Fourier series. Furthermore, let G_n be the corresponding Cesàro mean; that is,

$$G_n(x) = \frac{1}{n+1} \sum_{k=0}^n S_k(x) = p_0^n + \sum_{k=1}^n p_k^n \cos(2\pi kx) + q_k^n \sin(2\pi kx).$$

Then by Fejér's theorem (see Theorem 1.10 in [14]) we have that

$$\lim_{n \rightarrow \infty} \|G_n - G\|_{C^2(\mathbb{T})} = 0.$$

Next, G satisfies (2.1), (2.2) so $p_0 > 0$ and $p_k \geq 0$ for $k \geq 1$. Therefore, we have that

$$\begin{aligned} p_0^n &= p_0 > 0, \\ p_k^n &= \frac{n+1-k}{n+1} p_k \geq 0, \quad 1 \leq k \leq n, \end{aligned}$$

so G_n also satisfy (2.2), (2.1) for all $n \geq 1$.

Now, we can complete the proof of Theorem 1.7. We approximate G using Part 2 and conclude using Part 1. \square

6. NUMERICAL SOLUTIONS

Here, we numerically solve (1.6) for different types of kernels G . We present three cases. First, we consider G that is a non-symmetric trigonometric polynomial. Second, we consider G that is a symmetric trigonometric polynomial. And third, we consider G that is periodic but that is not a trigonometric polynomial.

During the whole discussion in this section we assume that

$$\begin{cases} V(x) = 2 \sin\left(2\pi\left(x + \frac{1}{4}\right)\right), & x \in \mathbb{T}, \\ \alpha = 1.5, & j = \sqrt{2}. \end{cases}$$

This choice of parameters in (1.6) is random and robustness of our calculations does not depend on a particular choice of parameters.

6.1. The case of a non-symmetric trigonometric polynomial. By Theorem 1.3 we have that for a given non-symmetric trigonometric polynomial G the solution m of (1.6) has the form (1.10), where the vector $(a_0^*, a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*)$ is the unique solution of (1.12). Furthermore, we define

$$\begin{aligned} M(\mathbf{a}, \mathbf{b}) = & \left(\frac{\partial \Phi_\alpha(\mathbf{a}, \mathbf{b})}{\partial a_0} - 1 \right)^2 + \frac{1}{2} \sum_{k=1}^n \left(\frac{\partial \Phi_\alpha(\mathbf{a}, \mathbf{b})}{\partial a_k} - \frac{p_k}{p_k^2 + q_k^2} a_k - \frac{q_k}{p_k^2 + q_k^2} b_k \right)^2 \\ & + \frac{1}{2} \sum_{k=1}^n \left(\frac{\partial \Phi_\alpha(\mathbf{a}, \mathbf{b})}{\partial b_k} - \frac{p_k}{p_k^2 + q_k^2} b_k + \frac{q_k}{p_k^2 + q_k^2} a_k \right)^2, \end{aligned}$$

where $(\mathbf{a}, \mathbf{b}) = (a_0, a_1, \dots, a_n, b_1, \dots, b_n) \in \mathcal{C}$. Then, solutions of (1.12) coincide with minimums of M . Accordingly, we find the solution of (1.12) by numerically solving the optimization problem

$$\min_{(\mathbf{a}, \mathbf{b}) \in \mathcal{C}} M(\mathbf{a}, \mathbf{b}). \quad (6.1)$$

We devise our algorithm in Wolfram Mathematica[®] language and use the built-in optimization function `FindMinimum` to solve (6.1).

As an example, we consider the kernel

$$G_1(x) = 1 + 4 \cos(2\pi x) - 5 \sin(2\pi x) + \cos(4\pi x) - 2 \sin(4\pi x), \quad x \in \mathbb{T}.$$

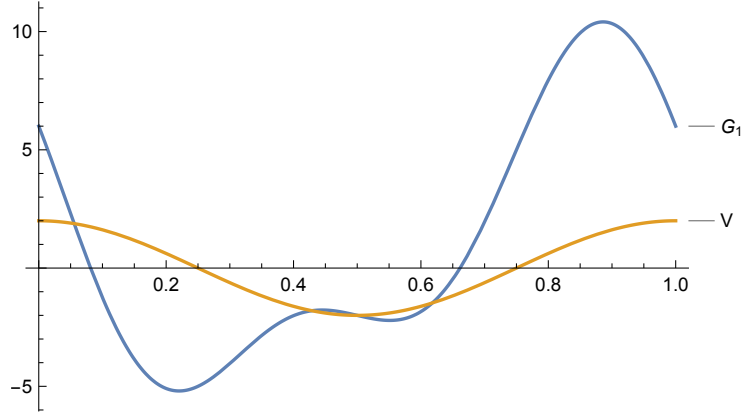
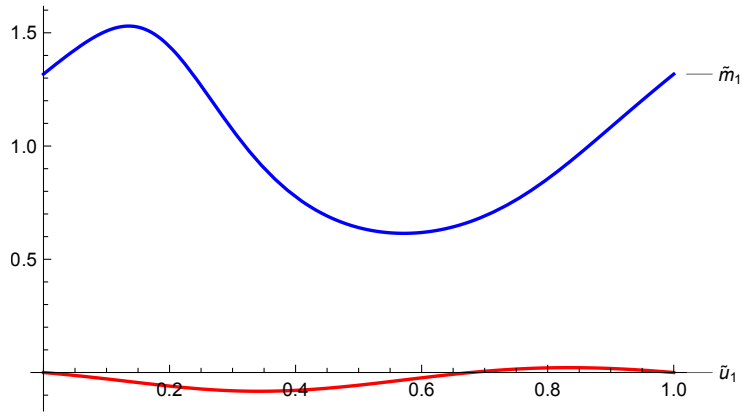
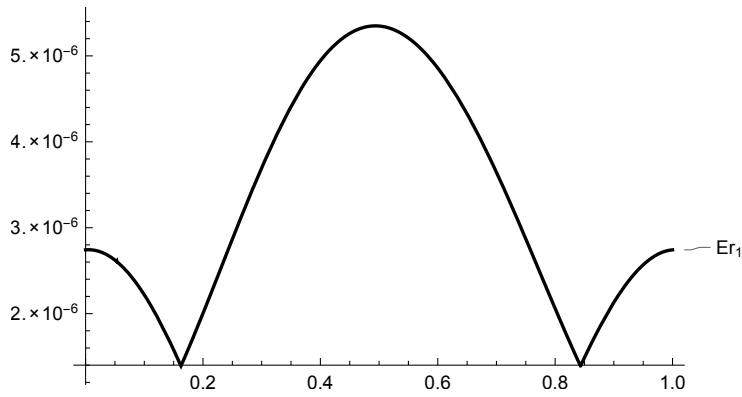
We denote by $(\tilde{u}_1, \tilde{m}_1, \tilde{H}_1)$ the corresponding numerical solution of (1.1). We first find $(\tilde{m}_1, \tilde{H}_1)$ by solving (6.1) and using (1.10) and (1.11). Next, we use (1.5) to find \tilde{u}_1 .

Finally, to estimate the accuracy of numerical solutions we introduce the error function

$$\begin{aligned} \text{Er}_1(x) = & \left| \frac{j^2}{2\tilde{m}_1(x)^\alpha} + V(x) - \int_{\mathbb{T}} G_1(x-y) \tilde{m}_1(y) dy - \tilde{H}_1 \right| \\ & + \left| \int_{\mathbb{T}} \tilde{m}_1(y) dy - 1 \right|, \quad x \in \mathbb{T}. \end{aligned}$$

We plot G_1 and V in Fig. 1, \tilde{m}_1 and \tilde{u}_1 in Fig. 2 and Er_1 in Fig. 3.

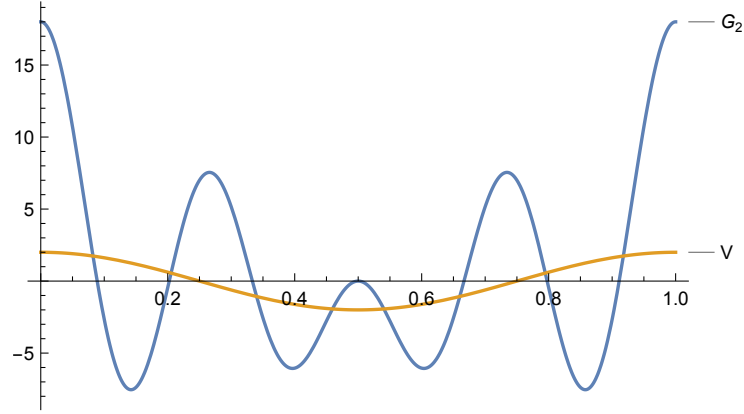
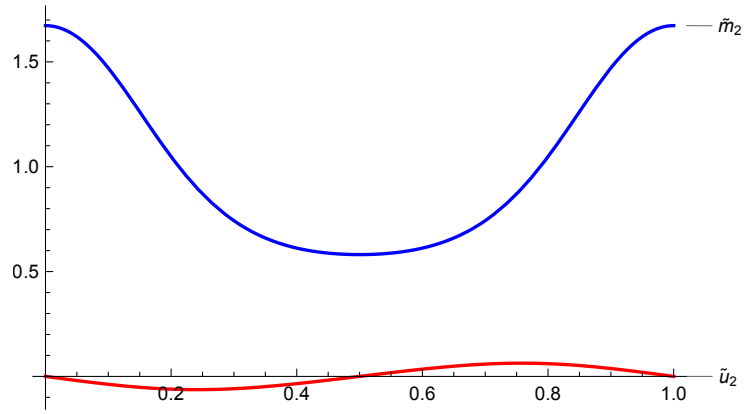
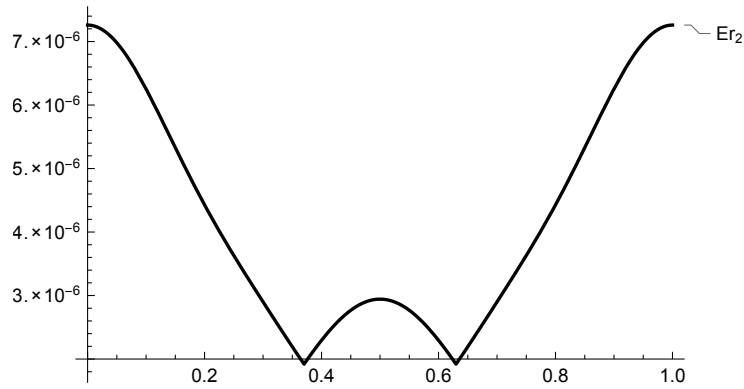
6.2. The case of a symmetric trigonometric polynomial. By Corollary 1.5 we have that for a given symmetric trigonometric polynomial G the solution m of (1.6) has the form (1.10), where the vector $(a_0^*, a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*)$ is the unique solution of (1.14). As before, we use `FindMinimum` to solve (1.14).

FIG. 1. The kernel G_1 and the potential V .FIG. 2. The approximate solutions \tilde{m}_1 and \tilde{u}_1 .FIG. 3. The error Er_1 .

As an example, we consider the kernel

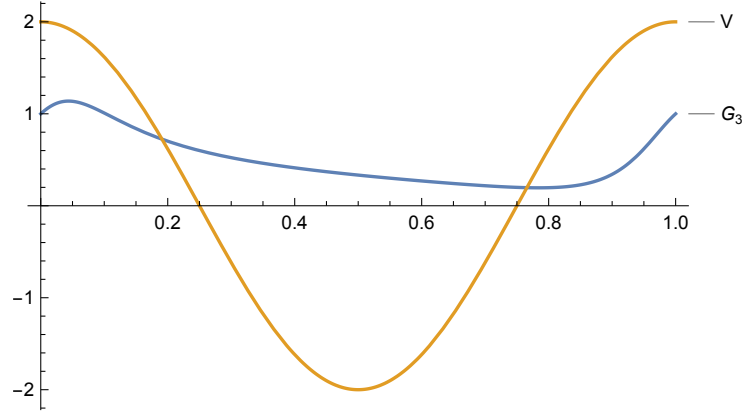
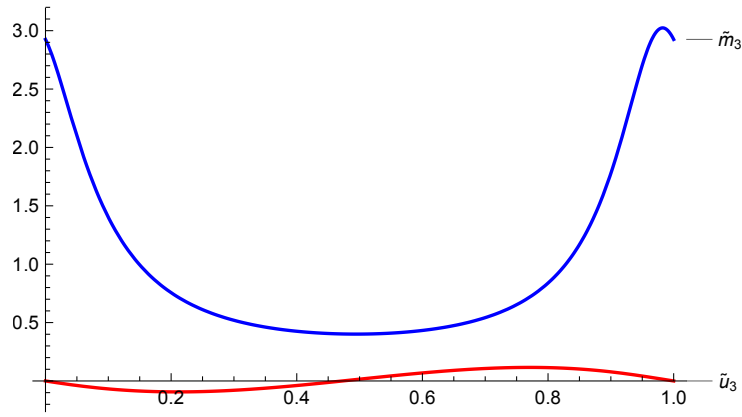
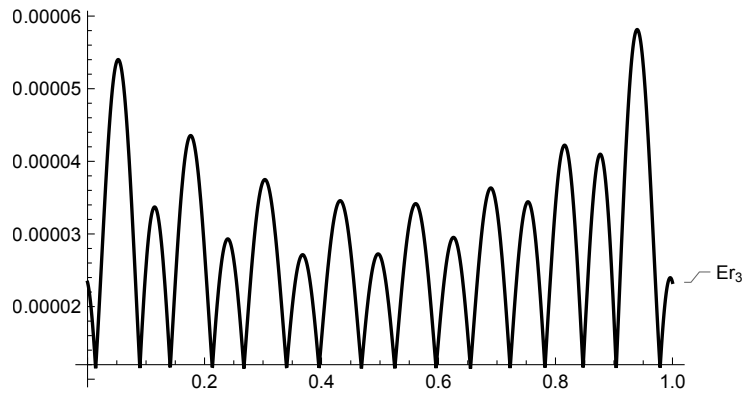
$$G_2(x) = 1 + 4 \cos(2\pi x) + \cos(4\pi x) + 5 \cos(6\pi x) + 7 \cos(8\pi x), \quad x \in \mathbb{T}.$$

Analogous to the previous case we denote by $(\tilde{u}_2, \tilde{m}_2, \tilde{H}_2)$ the numerical solution of (1.1) corresponding to G_2 . Furthermore, we denote by Er_2 the error function corresponding to $(\tilde{u}_2, \tilde{m}_2, \tilde{H}_2)$. We plot G_2 and V in Fig. 4, \tilde{m}_2 and \tilde{u}_2 in Fig. 5 and Er_2 in Fig. 6.

FIG. 4. The kernel G_2 and the potential V .FIG. 5. The approximate solutions \tilde{m}_2 and \tilde{u}_2 .FIG. 6. The error Er_2 .

6.3. The case of a non trigonometric polynomial. If G is not a trigonometric polynomial we first approximate it by its truncated Fourier series and then apply one of the previous solution methods. As an example we take

$$G_3(x) = \frac{2 - \cos(2\pi x) + \sin(2\pi x)}{5 - 4 \cos(2\pi x)}, \quad x \in \mathbb{T}.$$

FIG. 7. The kernel G_3 and the potential V .FIG. 8. The approximate solutions \tilde{m}_3 and \tilde{u}_3 .FIG. 9. The error Er_3 .

As before, we denote by $(\tilde{u}_3, \tilde{m}_3, \tilde{H}_3)$ and Er_3 the numerical solution of (1.1) and the error function corresponding to G_3 , respectively. We plot G_3 and V in Fig. 7, \tilde{m}_3 and \tilde{u}_3 in Fig. 8 and Er_3 in Fig. 9.

7. EXTENSIONS

Here, we discuss how our methods can be applied to other one-dimensional MFG system such as (1.19). Denote by $L : \mathbb{T} \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$, $(x, v, m) \mapsto L(x, v, m)$, be

the Legendre transform of H ; that is,

$$L(x, v, m) = \sup_{p \in \mathbb{R}} (vp - H(x, p, m)).$$

Then, if H satisfies suitable conditions, we have that

$$L(x, v, m) + H(x, p, m) \geq vp, \quad (7.1)$$

for all $v, p \in \mathbb{R}$ and there is equality in (7.1) if and only if

$$v = H'_p(x, p, m) \quad \text{or} \quad p = L'_v(x, v, m). \quad (7.2)$$

As before, second equation in (1.19) yields

$$H'_p(x, u_x, m) = \frac{j}{m},$$

for some constant j . Therefore, using (7.2) we find

$$u_x = L'_v\left(x, \frac{j}{m}, m\right),$$

which we plug-in to the first equation in (1.19) and obtain the following system

$$\begin{cases} H\left(x, L'_v\left(x, \frac{j}{m}, m\right), m\right) = \mathcal{F}\left(\int_{\mathbb{T}} G(x-y)m(y)dy\right) + \overline{H}, \\ m > 0, \int_{\mathbb{T}} m(x)dx = 1. \end{cases} \quad (7.3)$$

Next, one can attempt to study (7.3) first when G is a trigonometric polynomial and then approximate the general case. As before, when G is a trigonometric polynomial the expression

$$\int_{\mathbb{T}} G(x-y)m(y)dy$$

is always a trigonometric polynomial. Therefore, we have that

$$H\left(x, L'_v\left(x, \frac{j}{m}, m\right), m\right) = \mathcal{F}\left(\sum_{k=0}^n a_k^* \cos(2\pi kx) + b_k^* \sin(2\pi kx)\right) + \overline{H}, \quad (7.4)$$

for some $\{a_k^*\}, \{b_k^*\} \subset \mathbb{R}$. Suppose H is such that the left-hand-side expression of (7.4) is invertible in m with inverse $A_j(x, m)$. Then, (7.4) yields the following ansatz

$$m(x) = A_j\left(x, \mathcal{F}\left(\sum_{k=0}^n a_k^* \cos(2\pi kx) + b_k^* \sin(2\pi kx)\right) + \overline{H}\right). \quad (7.5)$$

Thus, one can search for the solution m of (7.3) in the form (7.5) with undetermined coefficients $\{a_k^*\}, \{b_k^*\}, \overline{H}$. Therefore, by plugging (7.5) in (7.3) we obtain a finite-dimensional fixed point problem for $\{a_k^*\}, \{b_k^*\}, \overline{H}$. If this fixed point problem has good structural properties (such as (1.12)) for a concrete model of the form (1.19), one may analyze this model by methods developed here.

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